Open Problems in Complex Analysis and Dynamical Systems

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Abstracts

Minimal area problems and its connection with quadrature domains

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Abstract

The tool of quadrature domains was developed by Harold Shapiro and myself many years ago in order to attack minimal area peoblems in the area of geometric function theory. Inspite of the great progress made during the years, the minimal area problem in its full generality, is still wide open and considered very deep. In this talk we survey the main obstacles and explain why the method used to solve some special cases can not be generalized.

Geometric generalizations in Kresin-Maz'ya sharp real-part theorems

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Abstract

We present geometric generalizations of the estimates given in the recent research book of Kresin and Maz'ya *"Sharp Real-Part Theorems. A Unified Approach"*. These estimates allow us to extend their sharpness to new cases. This is a joint work with Alekos Vidras, Nicosia, Cyprus.

Boundary interpolation and rigidity theorems

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Abstract

A generalized Schur function can be defined as a quotient of a function analytic and contractive in the open unit disk with a finite Blaschke product. An indirect way to define a generalized Nevanlinna function N is to say that $(1 + iN)/(1 - iN)$ is a generalized Schur function. We discuss the boundary interpolation problem for these functions and as application present new rigidity theorems. The talk is based on various works with A. Dijksma, H. Langer, D. Shoikhet and S. Reich.

Dispersive regularization and the Hele-Shaw problem

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Abstract

The relation between the Hele-Shaw problem and integrable systems is discussed, stressing the relation to the dispersive regularization approach. The dynamics of the Hele-Shaw problem is connected to the dynamics of the spectral surface dictated by Whitham's equations. The relation allows to find solutions for the Hele-Shaw problem without having to find a uniformizing conformal map.

Loewner equations on hyperbolic manifolds

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Abstract

The classical Loewner equations relate on one side a Loewner chain of univalent functions fixing the origin to a non-autonomous holomorphic vector field and on the other side the same vector field to a so-called evolution family of holomorphic self-maps of the unit disc fixing the origin. Such objects can be viewed as a generalization of discrete iteration and continuous semigroups of holomorphic maps. In this talk I will describe a general strategy, developed with M. Contreras and S. Diaz-Madrigal, to define and solve the vector field/evolution family differential equation on complete hyperbolic complex manifolds. In particular this point of view does not require any fixed point conditions on the evolution family and does not use at all distorsion theorems (which are not in general available in higher dimension) but relies on intrinsic properties of the Kobayashi metric.

Directional dilatations in space

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Abstract

The classical Grötzsch estimate of distortion of the circular rings $\{r \leq |z| \leq 1\}$ and $\{\rho \leq |w| \leq 1\}$ yields that by a q-quasiconformal mapping $z \mapsto w$,

$$r^q \leq \rho \leq r^{\frac{1}{q}}.$$

In [2], the authors have established the inequalities of the Grötzsch type, which estimate the distortion of conformal moduli of the ring domains under quasiregular mappings f in $\mathbb{R}^n$, $n \geq 2$. In particular, for the spherical rings $R(a,b) = \{x : a < |x| < b\}$ and $R(c,d) = \{y : c < |y| < d\}$,

$$\left|\log \frac{d}{c} - \log \frac{b}{a}\right| \leq \frac{1}{\omega_{n-1}} \int\limits_{R(a,b)} \frac{L_f(x) - 1}{|x|^n} dx.$$

Here $L_f(x)$ denotes the inner dilatation of f at x.

In this talk, we consider more general dilatations, which include the tangential and radial ones. In the case of plane, those were introduced in [1] and [3], respectively. We apply these dilatations to establishing differential and geometric properties of generalized quasiconformal mappings in $\mathbb{R}^n$.

References

[1] C. Andreian Cazacu, *Sur les transformations pseudo-analytiques*, Revue Math. Pures Appl., **2** (1957), 383–397.

[2] Ch. Bishop, V. Ya. Gutlyanskiĭ, O. Martio, M. Vuorinen, *On conformal dilatation in space*, Int. J. Math. Math. Sci. **22** (2003), 1397–1420.

[3] O. Martio, V. Ryazanov, U. Srebro, E. Yakubov, *On ring solutions of Beltrami equations*, J. Analyse Math., **96** (2005), 117–150.

Relations between the properties of a Dynamical System and the topology of certain appropriate Inverse Limit of that Dynamical System

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Abstract

For a sequence $S = S_0 \xleftarrow{f_0} S_1 \xleftarrow{f_1} S_2$ ... of Riemann Surfaces S_i and branched coverings $f_i : S_{i+1} \rightarrow S_i$ of finite degrees $d_i > 1$ we define a Plaque Inverse Limit S_∞ as the set of all sequences of points $x = x_0 \in S_0, x_1 \in S_1, ...$ such that $f_{i+1}(x_{i+1}) = x_i$ equipped with the following topology [here our definition differs from the standard one - Plaque Inverse Limit has more open sets]: The base of the topology of S_∞ consists of all sequences of open sets $U = U_0 \subset S_0, U_1 \subset S_1, ...$ such that $f_i(U_{i+1}) = U_i$ [unlike Tychonoff topology of the standard Inverse Limit which requires for every sequence U to exist n such that $U_{i+1} = f_i^{-1}(U_i)$ for all $i > n$]. Plaque inverse limit S_∞ is naturally equipped with continuous covering maps $p_i : S_\infty \rightarrow S_i$ such that $f_i \circ p_{i+1} = p_i$.

Note that the local base of topology at $x \in S_\infty$ consists of all open sets U in S_∞ containing x such that each U_i is conformally equivalent to the unit disk in $\mathcal{C}$ and f_i restricted to U_{i+1} is conformally equivalent to some z^d where $1 \leq d \leq d_i$. Such open sets are called plaques. We will always assume that the open neighborhoods of point x are of that type. A point in S_∞ is called regular if for some its open neighborhood U exists n such that U_{i+1} contains no critical points of f_i for all $i \geq n$ [and so the degree of f_i restricted to U_{i+1} is 1]. Otherwise a point in S_∞ is called irregular. The set Δ of all regular points in S_∞ is clearly open and each of its connected components has a structure of a Riemann Surface. We prove that if all the degrees d_i are equal then Δ is dense in S_∞ but if $d_{i+1} \geq 2d_0d_1d_2...d_i$ for all i then Δ can be empty. We show that removing irregular points from their neighborhoods breaks these neighborhoods into uncountable amount of connected components disconnected from each-other. Thus the topology of Plaque inverse limit is of the special interest locally at the irregular points.

For each S_i we define the thickening S'_i as the set of all converging sequences of points in S_i and we equip S'_i with topology by setting its local base at point $p = p_1, p_2, ...$ to be the set all sequences of open neighborhoods of points $p1, p_2, ...$ in S_i. Since all f_i are continuous open maps and so both images and pre-images of converging sequences are converging sequences we obtain the thickening S' of system S. We define the thickening S'_∞ as the Plaque inverse limit of S'. Note that there is a natural imbedding of S_i into S'_i and of S_∞ into S'_∞ and that the lim map [which corresponds to each converging sequence its limit] is its left inverse. Using the concept of thickening we define continuous structure-preserving maps between plaque inverse limits of systems.

If $S_0 = S_1 = ...$ and $f_0 = f_1 = ...$ we obtain a dynamical system. The Riemann Surface structure of the connected components of Δ for this case is described in "Laminations in Holomorphic Dynamics" by Lyubich and Minsky. In this work we develop a new σ-Algebraic type machinery to fully describe and classify the Topology of S_∞ for the special case of all $S_i = \mathcal{C}$ and all $f_i = z^d + c$ as follows: Let $I = \{F, T\}^{\aleph_0}$ be the set of all countable sequences of F and T. I is a Boolean Algebra $(0 = \{F, F, ...\}, 1 = \{T, T, ...\} \in I$ and $\vee, \wedge, \neg$ are performed at each position). For all $x, y \in I$ we say that they are equivalent $[x \sim y]$ if x and y differ only in finitely many positions. The Boolean Algebraic structure of I then pushes down to $I/\sim$. Let A be the set of all subsets $\alpha \subset I/\sim$ such that if $[y] \in \alpha$ then $[x] \in \alpha$ for all $[x] \in I/\sim$ where $[x] < [y]$. We develop σ-Algebraic type machinery in A. Next, to every point of S_∞ we associate an element of A which is called the signature of that point. We show that the signature is a local Topological Invariant of S_∞ and that the signature of a point is $\{[0]\}$ if and only if this point is regular. We demonstrate some strong relations between signatures of irregular points and the Dynamical Properties of the System - for example we prove that every case of a map with attracting periodic cycle will have its own unique signatures at its irregular points and we compute all these signatures.

Quasiconformally invariant cohomology

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Abstract

One of the important questions in the theory of quasiconformal mappings is a construction of quasiconformal invariants of Riemannian manifolds which are stable under quasiconformal homeomorphisms and can be used to distinguish non quasiconformally equivalent manifolds. We describe a version of $L_{q,p}$-cohomology adopted to quasiconformal (conformal) geometry. We call these cohomology the conformal cohomology. The corresponding de Rham complex is a Banach differential graded algebra which id invariant under quasiconformal homeomorphisms.

We give examples of calculations of these conformal cohomology. As an application we prove that 3-dimensional Lie group SOL is not quasiconformally equivalent to the 3-dimensional hyperbolic space. The result can not be proved by the standard methods based on capacities because both spaces are homogeneous spaces with exponential volume growth.

Potentially interesting applications of the conformal cohomology will be discussed.

The Königs Problem and Extreme Fixed Points

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Abstract

The authors continue their investigation. An affine f.l.m. $\mathcal{F}_A : \mathcal{K} \to \mathcal{K}$ of the unit operator-valued ball is considered in the case where the fixed point C of the continuation of $\mathcal{F}_A$ to $\overline{\mathcal{K}}$ is either an isometry or a coisometry. For the case in which one of the diagonal elements (for example, A_{11}) of the operator matrix A is identical, the structure of the other diagonal element (A_{22}) is studied completely. It is shown that, in all these reasonings, C cannot be replaced by an arbitrary point of the unit sphere; some special cases in which this is still possible are studied. In conclusion, the KE-property of the mapping $\mathcal{F}_A$ is proved. It is joint work with V. Senderov

Irreducibility of dynamics and representation of KMS states in terms of Lèvy processes

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Abstract

A C*-dynamical system, where the group of time automorphisms is generated by a pseudo-differential operator of a certain kind is considered. It is proven that the subalgebra generated by the trajectories starting from a "sufficiently rich" family of mutually commuting elements is dense in the whole algebra in the σ-weak topology. This result allows for representing σ-weakly continuous states in terms of path integrals against measures associated with Lèvy processes. In particular, this leads to the representation of KMS states of systems of weakly relativistic quantum particles in terms of such integrals.

The dilatation function of a holomorphic isotopy

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Abstract

Every non vanishing univalent function $f(z)$ in the disk $\Delta^* = \widehat{\mathbf{C}} \setminus \overline{\Delta}$, $\Delta = \{|z| < 1\}$, for example, with hydrodynamical normalization, generates a complex isotopy $f_t(z) = tf(t^{-1}z) : \Delta^* \times \Delta \to \widehat{\mathbf{C}}$, which is a special case of holomorphic motions and plays an important role in many subjects areas. Let q_f denote the minimal dilatation among quasiconformal extensions of f to $\widehat{\mathbf{C}}$.

In 1995, R. Kühnau raised the questions whether the dilatation function $q_f(r) = q_{f_r}$ is real analytic and whether the function f can be reconstructed if $q_f(r)$ is given. The analyticity of q_f was known only for ellipses and for the Cassini ovals.

These intriguing problems still remain open. Note that the general results on the smoothness of Teichmüller distance provide that this distance at generic points is at most C^2 smooth.

Our main theorem provides a wide class of maps (containing, in particular, those whose extremal quasiconformal extensions from Δ^* across the unit circle are defined by holomorphic quadratic differentials with zeros of even order), which possess the analytic dilatation functions. It also implies a negative answer to the second question.

The proof involves the Grunsky coefficient inequalities technique for univalent functions with quasiconformal extensions and certain deep results of complex geometry of the universal Teichmüller space. It reduces to the construction and comparison of metrics with appropriate curvature properties.

Holomorphic retracts

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Abstract

In this talk we discuss the problem of intersection of infinitely many holomorphic retracts. This is a joint work with Monika Budzyn'ska and Simeon Reich.

Hausdorff summability of power series

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Abstract

An overview of the results on Hausdorff summability of power series of functions from the Hardy spaces will be given. All these results are very recent. The main tool for obtaining such results is composition operators. Certain open problems will be posed.

New results in physics of Laplacian growth

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Abstract

Recent experimental discoveries in the Hele-Shaw cell are reported and analyzed. A new class of exact solutions of the 2D Laplacian growth is presented to explain the experiments. These solutions, which describe arbitrary cuts dynamics, include all known exact solutions for Abelian domains.

About some unsolved problems in the theory of elliptic functional differential equations

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Abstract

1. Spectrum of differential-difference operator. Let $d = N+\theta$, $N \in \mathbb{N}$, $0 < \theta \leqslant 1$, and $a_{-N}, \dots, a_N$ be complex numbers. Given the non-degenerate differential-difference operator $A : L_2(0,d) \to L_2(0,d)$,

$$Au = -\left(\sum_{k=-N}^{N} a_j u(x+k)\right)'', \quad D(A_R) = \{u \in \mathring{W}_2^1(0,d) : Au \in L_2(0,d)\},$$

to find out whether its spectrum $\sigma(A)$ is discrete or fill up the complex plane.

Direct calculations show the discreteness of $\sigma(A)$ for small N. For N arbitrary, the discreteness is proved only under some additional restrictions such as the symmetry of the difference operator or the positivity of its symmetrical part in $L_2(0,d)$.

2. Gårding-type inequality for functional differential operators. (a) Let $Q \subset \mathbb{R}^n$ be a bounded domain. Consider the differential-difference equation

$$A_R u \equiv -\sum_{i,j=1}^{n} (R_{ij} u_{x_j})_{x_i} = f(x) \quad (x \in Q), \tag{0.1}$$

where $R_{ij}v(x) = \sum_h a_{ijh}(x)v(x+h)$, a_{ijh} are smooth complex-valued functions, and h runs over a finite set of vectors with integer coordinates (the function u is extended by zero to $\mathbb{R}^n$). A.L. Skubachevskii established algebraic sufficient conditions as well as necessary conditions for equation (0.1) to satisfy the estimate

$$\operatorname{Re}(A_R u, u)_{L_2(Q)} \geqslant c_1 \|u\|^2_{W_2^1(Q)} - c_2 \|u\|^2_{L_2(Q)} \quad (u \in \dot{C}^\infty(Q)). \tag{0.2}$$

For many Q, these sufficient conditions and necessary conditions coincide. Do they coincide for every bounded Q?

(b) Suppose that a bounded domain Q is such that $\overline{Q} \subset qQ$, $q > 1$, and the operators R_{ij} in (0.1) are $R_{ij}v(x) = \sum_k a_{ijk}(x)v(q^{-k}x)$, where k runs over a finite set of integers. The necessary and sufficient condition $\operatorname{Re} \sum a_{ijk} q^{kn/2} e^{\sqrt{-1}k\eta} \xi_i \xi_j > 0 \quad (\eta \in \mathbb{R},\ 0 \neq \xi \in \mathbb{R}^n)$ of estimate (0.2) is obtained in the case of constant coefficients; however, it is difficult to imagine what it can look like for variable coefficients. On the one hand, the standard localization technique does not work here. On the other hand, the points of Q are not equal, 0 stands out.

3. Smoothness of generalized solutions. It is known that the smoothness of generalized solutions to the first boundary value problem for strongly elliptic (i.e. satisfying estimate (0.2)) differential-difference equation (0.1) is disturbed in Q but is preserved in some subdomains Q_r such that $\cup \overline{Q}_r = \overline{Q}$. Investigate the same phenomenon for strongly elliptic functional differential equation with contractions and expansions. When $0 \in Q$, even the local smoothness in subdomains is not proven.

Weighted exponential approximation

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Abstract

I am going to discuss several recent results on the stability of the weighted exponential approximation under perturbations of the weight function, and their relation to the inverse spectral problem for Sturm-Liouville operators on a finite interval. This is a joint work with A. Borichev (Marseille).